

Some Operators Associated to Rarita-Schwinger Type Operators

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Abstract

In this paper we study some operators associated to the Rarita-Schwinger operators. They arise from the difference between the Dirac operator and the Rarita-Schwinger operators. These operators are called remaining operators. They are based on the Dirac operator and projection operators $I - P_k$. The fundamental solutions of these operators are harmonic polynomials, homogeneous of degree k . First we study the remaining operators and their representation theory in Euclidean space. Second, we can extend the remaining operators in Euclidean space to the sphere under the Cayley transformation.

Keywords: Clifford algebra, Rarita-Schwinger operators, remaining operators, Cayley transformation, Almansi-Fischer decomposition.

This paper is dedicated to Michael Shapiro on the occasion of his 65th birthday.

1 Introduction

Rarita-Schwinger operators in Clifford analysis arise in representation theory for the covering groups of $SO(n)$ and $O(n)$. They are generalizations of the Dirac operator. We denote a Rarita-Schwinger operator by R_k , where $k = 0, 1, \dots, m, \dots$. When $k = 0$ it is the Dirac operator. The Rarita-Schwinger operators R_k in Euclidean space have been studied in [BSSV, BSSV1, DLRV, Va1, Va2]. Rarita-Schwinger operators on the sphere denoted by R_k^S have also been studied in [LRV].

In this paper we study the remaining operators, Q_k , which are related to the Rarita-Schwinger operators. In fact, The remaining operators are the difference between the Dirac operator and the Rarita-Schwinger operators.

Let \mathcal{H}_k be the space of harmonic polynomials homogeneous of degree k and $\mathcal{M}_k, \mathcal{M}_{k-1}$ be the spaces of Cl_n -valued monogenic polynomials, homogeneous of degree k and $k-1$ respectively. Instead of considering $P_k : \mathcal{H}_k \rightarrow \mathcal{M}_k$ in [DLRV], we look at the projection map, $I - P_k : \mathcal{H}_k \rightarrow u\mathcal{M}_{k-1}$, and the Dirac operator, to define the Q_k operators and construct their fundamental solutions in \mathbb{R}^n . We introduce basic results of these operators. This includes Stokes' Theorem, Borel-Pompeiu Theorem, Cauchy's Integral Formula and a Cauchy Transform. In section 5, by considering the Cayley transformation and its

inverse, we can extend the results for the remaining operators in \mathbb{R}^n to the sphere, \mathbb{S}^n . We construct the fundamental solutions to the remaining operators by applying the Cayley transformation to the fundamental solutions to the remaining operators in \mathbb{R}^n . We also obtain the intertwining operators for Q_k operators and Q_k^S operators. In turn, we establish the conformal invariance of the remaining equations under the Cayley transformation and its inverse. We conclude by giving some basic integral formulas with detailed proofs, and pointing out that results obtained for Rarita-Schwinger operators in [LRV] for real projective space readily carry over to the context presented here.

2 Preliminaries

A Clifford algebra, Cl_n , can be generated from \mathbb{R}^n by considering the relationship

$$x^2 = -\|x\|^2$$

for each $x \in \mathbb{R}^n$. We have $\mathbb{R}^n \subseteq Cl_n$. If e_1, \dots, e_n is an orthonormal basis for \mathbb{R}^n , then $x^2 = -\|x\|^2$ tells us that $e_i e_j + e_j e_i = -2\delta_{ij}$, where δ_{ij} is the Kronecker delta function. Let $A = \{j_1, \dots, j_r\} \subset \{1, 2, \dots, n\}$ and $1 \leq j_1 < j_2 < \dots < j_r \leq n$. An arbitrary element of the basis of the Clifford algebra can be written as $e_A = e_{j_1} \cdots e_{j_r}$. Hence for any element $a \in Cl_n$, we have $a = \sum_A a_A e_A$, where $a_A \in \mathbb{R}$.

We define the Clifford conjugation as the following:

$$\bar{a} = \sum_A (-1)^{|A|(|A|+1)/2} a_A e_A$$

satisfying $\overline{e_{j_1} \cdots e_{j_r}} = (-1)^r e_{j_r} \cdots e_{j_1}$ and $\overline{ab} = \bar{b} \bar{a}$ for $a, b \in Cl_n$.

For each $a = a_0 + \dots + a_{1\dots n} e_1 \cdots e_n \in Cl_n$ the scalar part of $\bar{a}a$ gives the square of the norm of a , namely $a_0^2 + \dots + a_{1\dots n}^2$.

The reversion is given by

$$\tilde{a} = \sum_A (-1)^{|A|(|A|-1)/2} a_A e_A,$$

where $|A|$ is the cardinality of A . In particular, $\widetilde{e_{j_1} \cdots e_{j_r}} = e_{j_r} \cdots e_{j_1}$. Also $\tilde{a}b = \tilde{b}a$ for $a, b \in Cl_{n+1}$.

The Pin and Spin groups play an important role in Clifford analysis. The Pin group can be defined as

$$Pin(n) := \{a \in Cl_n : a = y_1 \dots y_p : y_1, \dots, y_p \in \mathbb{S}^{n-1}, p \in \mathbb{N}\}$$

and is clearly a group under multiplication in Cl_n .

Now suppose that $y \in \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. Look at $xyy = yx^{\parallel y}y + yx^{\perp y}y = -x^{\parallel y} + x^{\perp y}$ where $x^{\parallel y}$ is the projection of x onto y and $x^{\perp y}$ is perpendicular to y . So xyy gives a reflection of x in the y direction. By the Cartan–Dieudonné Theorem each $O \in O(n)$ is the composition of a finite number of reflections. If $a = y_1 \dots y_p \in Pin(n)$, then $\tilde{a} := y_p \dots y_1$ and $a\tilde{a} = O_a(x)$ for some $O_a \in O(n)$. Choosing y_1, \dots, y_p arbitrarily in \mathbb{S}^{n-1} , we see that the group homomorphism

$$\theta : Pin(n) \longrightarrow O(n) : a \longmapsto O_a$$

with $a = y_1 \dots y_p$ and $O_a(x) = ax\tilde{a}$ is surjective. Further $-ax(-\tilde{a}) = ax\tilde{a}$, so $1, -1 \in \ker(\theta)$. In fact $\ker(\theta) = \{\pm 1\}$. The Spin group is defined as

$$\text{Spin}(n) := \{a \in \text{Pin}(n) : a = y_1 \dots y_p \text{ and } p \text{ even}\}$$

and is a subgroup of $\text{Pin}(n)$. There is a group homomorphism

$$\theta : \text{Spin}(n) \longrightarrow \text{SO}(n)$$

which is surjective with kernel $\{1, -1\}$. See [P] for details.

The Dirac Operator in \mathbb{R}^n is defined to be

$$D := \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}.$$

Note $D^2 = -\Delta_n$, where Δ_n is the Laplacian in \mathbb{R}^n .

Let \mathcal{H}_k be the space of harmonic polynomials homogeneous of degree k . Let \mathcal{M}_k denote the space of Cl_n -valued polynomials, homogeneous of degree k and such that if $p_k \in \mathcal{M}_k$ then $Dp_k = 0$. Such a polynomial is called a left monogenic polynomial homogeneous of degree k . Note if $h_k \in \mathcal{H}_k$, the space of Cl_n -valued harmonic polynomials homogeneous of degree k , then $Dh_k \in \mathcal{M}_{k-1}$. But $Dup_{k-1}(u) = (-n - 2k + 2)p_{k-1}(u)$, so

$$\mathcal{H}_k = \mathcal{M}_k \bigoplus u\mathcal{M}_{k-1}, h_k = p_k + up_{k-1}.$$

This is the so-called Almansi-Fischer decomposition of \mathcal{H}_k , where \mathcal{M}_{k-1} is the space of Cl_n -valued left monogenic polynomials, homogeneous of degree $k-1$. See [BDS, R].

Note that if $Dg(u) = 0$ then $\bar{g}(u)\bar{D} = -\bar{g}(u)D = 0$. So we can talk of right monogenic polynomials, homogeneous of degree k and we obtain by conjugation a right Almansi-Fisher decomposition,

$$\mathcal{H}_k = \overline{\mathcal{M}_k} \bigoplus \overline{\mathcal{M}_{k-1}}u,$$

where $\overline{\mathcal{M}_k}$ stands for the space of right monogenic polynomials homogeneous of degree k .

Let P_k be the left projection map

$$P_k : \mathcal{H}_k \rightarrow \mathcal{M}_k,$$

then the left Rarita-Schwinger operator R_k is defined by (see [BSSV, BSSV1, DLRV, Va1, Va2])

$$R_k g(x, u) = P_k D_x g(x, u),$$

where D_x is the Dirac operator with respect to x and $g(x, u) : U \times \mathbb{R}^n \rightarrow Cl_n$ is a monogenic polynomial homogeneous of degree k in u , and U is a domain in \mathbb{R}^n . The left Rarita-Schwinger equation is defined to be

$$R_k g(x, u) = 0.$$

We also have a right projection $P_{k,r} : \mathcal{H}_k \rightarrow \overline{\mathcal{M}_k}$, and a right Rarita-Schwinger equation $g(x, u)D_x P_{k,r} = g(x, u)R_k = 0$.

A Möbius transformation is a finite composition of orthogonal transformations, inversions, dilations, and translations. Ahlfors [A] and Vahlen [V] show that given a Möbius transformation $y = \phi(x)$ on $\mathbb{R}^n \cup \{\infty\}$ it can be expressed as $y = (ax + b)(cx + d)^{-1}$ where $a, b, c, d \in Cl_n$ and satisfy the following conditions:

1. a, b, c, d are all products of vectors in \mathbb{R}^n .
2. $a\tilde{b}, c\tilde{d}, \tilde{b}c, \tilde{d}a \in \mathbb{R}^n$.
3. $a\tilde{d} - b\tilde{c} = \pm 1$.

When $c = 0$, $\phi(x) = (ax + b)(cx + d)^{-1} = axd^{-1} + bd^{-1} = \pm ax\tilde{a} + bd^{-1}$. Now assume $c \neq 0$, then $\phi(x) = (ax + b)(cx + d)^{-1} = ac^{-1} \pm (cx\tilde{c} + d\tilde{c})^{-1}$. These are the so-called Iwasawa decompositions. Using this notation and the conformal weights, $f(\phi(x))$ is changed to

$J(\phi, x)f(\phi(x))$, where $J(\phi, x) = \frac{\widetilde{cx + d}}{\|cx + d\|^n}$. Note when $\phi(x) = x + a$ then $J(\phi, x) \equiv 1$.

3 The Q_k operators and their kernels

As

$$I - P_k : \mathcal{H}_k \rightarrow u\mathcal{M}_{k-1},$$

where I is the identity map, then we can define the left remaining operators

$$Q_k := (I - P_k)D_x : u\mathcal{M}_{k-1} \rightarrow u\mathcal{M}_{k-1} \quad uf(x, u) \mapsto (I - P_k)D_x uf(x, u).$$

See[BSSV].

The left remaining equation is defined to be $(I - P_k)D_x uf(x, u) = 0$ or $Q_k uf(x, u) = 0$, for each x and $(x, u) \in U \times \mathbb{R}^n$, where U is a domain in \mathbb{R}^n and $f(x, u) \in \mathcal{M}_{k-1}$.

We also have a right remaining operator

$$Q_{k,r} := D_x(I - P_{k,r}) : \overline{\mathcal{M}}_{k-1}u \rightarrow \overline{\mathcal{M}}_{k-1}u \quad g(x, u)u \mapsto g(x, u)uD_x(I - P_{k,r}),$$

where $g(x, u) \in \overline{\mathcal{M}}_{k-1}$.

Consequently, the right remaining equation is $g(x, u)uD_x(I - P_{k,r}) = 0$ or $g(x, u)uQ_{k,r} = 0$.

Now let us establish the conformal invariance of the remaining equation $Q_k uf(x, u) = 0$.

It is easy to see that $I - P_k$ is conformally invariant under the Möbius transformations, since the projection operator P_k is conformally invariant (see [BSSV, DLRV]). By considering orthogonal transformation, inversion, dilation and translation and applying the same arguments in [DLRV] used to establish the intertwining operators for Rarita-Schwinger operators, we can easily obtain the intertwining operators for Q_k operators:

Theorem 1.

$$J_{-1}(\phi, x)Q_{k,u}uf(y, u) = Q_{k,w}wJ(\phi, x)f(\phi(x), \frac{\widetilde{(cx + d)w(cx + d)}}{\|cx + d\|^2}),$$

where $Q_{k,u}$ and $Q_{k,w}$ are the remaining operators with respect to u and w respectively,

$y = \phi(x)$ is the Möbius transformation, $J(\phi, x) = \frac{\widetilde{cx + d}}{\|cx + d\|^n}$, $J_{-1}(\phi, x) = \frac{cx + d}{\|cx + d\|^{n+2}}$,

and $u = \frac{\widetilde{(cx + d)w(cx + d)}}{\|cx + d\|^2}$ for some $w \in \mathbb{R}^n$.

Consequently, we have

$Q_{k,u}uf(x, u) = 0$ implies $Q_{k,w}wJ(\phi, x)f(\phi(x), \frac{\widetilde{(cx+d)w(cx+d)}}{\|cx+d\|^2}) = 0$. This tells us that the remaining equation $Q_kuf(x, u) = 0$ is conformally invariant under Möbius transformations.

The reproducing kernel of \mathcal{M}_k with respect to integration over \mathbb{S}^{n-1} is given by (see [BDS, DLRV])

$$Z_k(u, v) := \sum_{\sigma} P_{\sigma}(u) V_{\sigma}(v) v,$$

where

$$P_{\sigma}(u) = \frac{1}{k!} \Sigma(u_{i_1} - u_1 e_1^{-1} e_{i_1}) \dots (u_{i_k} - u_1 e_1^{-1} e_{i_k}), V_{\sigma}(v) = \frac{\partial^k G(v)}{\partial v_2^{j_2} \dots \partial v_n^{j_n}}$$

$j_2 + \dots + j_n = k$, and $i_k \in \{2, \dots, n\}$. Here summation is taken over all permutations of the monomials without repetition. This function is left monogenic in u and right monogenic polynomial in v and it is homogeneous of degree k . See [BDS] and elsewhere.

Let us consider the polynomial $uZ_{k-1}(u, v)v$ which is harmonic, homogeneous degree of k in both u and v . Since $uZ_{k-1}(u, v)v$ does not depend on x , $Q_k uZ_{k-1}(u, v)v = 0$.

Now applying inversion from the left, we obtain

$$H_k(x, u, v) := \frac{-1}{\omega_n c_k} u \frac{x}{\|x\|^n} Z_{k-1}\left(\frac{xux}{\|x\|^2}, v\right) v$$

is a non-trivial solution to $Q_k u f(x, u) = 0$, where $c_k = \frac{n-2}{n-2+2k}$.

Similarly, applying inversion from the right, we obtain

$$\frac{-1}{\omega_n c_k} u Z_{k-1}\left(u, \frac{v x v}{\|x\|^2}\right) \frac{x}{\|x\|^n} v$$

is a non-trivial solution to $f(x, v)vQ_{k,r} = 0$. Using the similar arguments in [DLRV], we can show that two representations of the solutions are equal. The details are given in the following

$$\begin{aligned} & \frac{-1}{\omega_n c_k} u Z_{k-1}\left(u, \frac{v x v}{\|x\|^2}\right) \frac{x}{\|x\|^n} v \\ &= \frac{-1}{\omega_n c_k} u \frac{-x}{\|x\|} Z_{k-1}\left(\frac{xux}{\|x\|^2}, v\right) \frac{x}{\|x\|} \frac{x}{\|x\|^n} v = \frac{-1}{\omega_n c_k} u \frac{x}{\|x\|^n} Z_{k-1}\left(\frac{xux}{\|x\|^2}, v\right) v. \end{aligned}$$

In fact $H_k(x, u, v)$ is the fundamental solution to the Q_k operator.

4 Some basic integral formulas related to Q_k operators

In this section, we will establish some basic integral formulas associated with Q_k operators.

Definition 1. [DLRV] For any Cl_n -valued polynomials $P(u), Q(u)$, the inner product $(P(u), Q(u))_u$ with respect to u is given by

$$(P(u), Q(u))_u = \int_{\mathbb{S}^{n-1}} P(u)Q(u)ds(u).$$

For any $p_k \in \mathcal{M}_k$, one obtains (see [BDS])

$$p_k(u) = (Z_k(u, v), p_k(v))_v = \int_{\mathbb{S}^{n-1}} Z_k(u, v)p_k(v)ds(v).$$

Now if we combine Stokes' Theorems of the Dirac operator and the Rarita-Schwinger operator, then we have two versions of Stokes' Theorem for the Q_k operators .

Theorem 2. (Stokes' Theorem for Q_k operators) Let Ω' and Ω be domains in \mathbb{R}^n and suppose the closure of Ω lies in Ω' . Further suppose the closure of Ω is compact and the boundary of Ω , $\partial\Omega$, is piecewise smooth. Then for $f, g \in C^1(\Omega', \mathcal{M}_k)$, we have version 1

$$\begin{aligned} & \int_{\Omega} [(g(x, u)Q_{k,r}, f(x, u))_u + (g(x, u), Q_k f(x, u))_u] dx^n \\ &= \int_{\partial\Omega} (g(x, u), (I - P_k)d\sigma_x f(x, u))_u \\ &= \int_{\partial\Omega} (g(x, u)d\sigma_x(I - P_{k,r}), f(x, u))_u. \end{aligned}$$

Then for $f, g \in C^1(\Omega', \mathcal{M}_{k-1})$, we have version 2

$$\begin{aligned} & \int_{\Omega} [(g(x, u)uQ_{k,r}, uf(x, u))_u + (g(x, u)u, Q_k uf(x, u))_u] dx^n \\ &= \int_{\partial\Omega} (g(x, u)u, (I - P_k)d\sigma_x uf(x, u))_u \\ &= \int_{\partial\Omega} (g(x, u)ud\sigma_x(I - P_{k,r}), uf(x, u))_u. \end{aligned}$$

Proof: It is easy to get version 1 of Stokes' Theorem for the Q_k operators by combining Stokes' Theorems of the Dirac operator and the Rarita-Schwinger operators.

Now we shall prove version 2 of Stokes' Theorem.

First of all, we want to prove that

$$\int_{\partial\Omega} (g(x, u)u, (I - P_k)d\sigma_x uf(x, u))_u = \int_{\partial\Omega} (g(x, u)ud\sigma_x(I - P_{k,r}), uf(x, u))_u.$$

Here $d\sigma_x = n(x)d\sigma(x)$. By the Almansi-Fischer decomposition, we have

$$g(x, u)un(x)uf(x, u) = g(x, u)u[f_1(x, u) + uf_2(x, u)] = [g_1(x, u) + g_2(x, u)u]uf(x, u),$$

so

$$\begin{aligned} & g(x, u)u d\sigma_x u f(x, u) \\ &= g(x, u)u[f_1(x, u) + u f_2(x, u)]d\sigma(x) = [g_1(x, u) + g_2(x, u)u]u f(x, u)d\sigma(x), \end{aligned}$$

where $f_1(x, u)$, $f_2(x, u)$, $g_1(x, u)$, $g_2(x, u)$ are left or right monogenic polynomials in u . Now integrating the above formula over the unit sphere in \mathbb{R}^n , one gets

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} g(x, u)u d\sigma_x u f(x, u)ds(u) \\ &= \int_{\mathbb{S}^{n-1}} g(x, u)u u f_2(x, u)d\sigma(x)ds(u) = \int_{\mathbb{S}^{n-1}} g_2(x, u)u u f(x, u)d\sigma(x)ds(u). \end{aligned}$$

This follows from the fact that

$$\int_{\mathbb{S}^{n-1}} g(x, u)u f_1(x, u)ds(u) = \int_{\mathbb{S}^{n-1}} g_1(x, u)u f(x, u)ds(u) = 0.$$

See [BDS].

Thus

$$\begin{aligned} \int_{\partial\Omega} (g(x, u)u, (I - P_k)d\sigma_x u f(x, u))_u &= \int_{\partial\Omega} \int_{\mathbb{S}^{n-1}} g(x, u)u((I - P_k)d\sigma_x u f(x, u))ds(u) \\ &= \int_{\partial\Omega} \int_{\mathbb{S}^{n-1}} g(x, u)u u f_2(x, u)ds(u)d\sigma(x) \\ &= \int_{\partial\Omega} \int_{\mathbb{S}^{n-1}} g_2(x, u)u u f(x, u)ds(u)d\sigma(x) \\ &= \int_{\partial\Omega} \int_{\mathbb{S}^{n-1}} (g(x, u)d\sigma_x(I - P_{k,r}))u f(x, u)ds(u) \\ &= \int_{\partial\Omega} (g(x, u)u d\sigma_x(I - P_{k,r}), u f(x, u))_u. \end{aligned}$$

Secondly, we need to show

$$\begin{aligned} & \int_{\Omega} [(g(x, u)u Q_{k,r}, u f(x, u))_u + (g(x, u)u, Q_k u f(x, u))_u]dx^n \\ &= \int_{\partial\Omega} (g(x, u)u, (I - P_k)d\sigma_x u f(x, u))_u. \end{aligned}$$

Consider the integral

$$\begin{aligned} & \int_{\Omega} [(g(x, u)u D_x P_{k,r}, u f(x, u))_u + (g(x, u)u, P_k D_x u f(x, u))_u]dx^n \\ &= \int_{\Omega} \int_{\mathbb{S}^{n-1}} [(g(x, u)u D_x P_{k,r})u f(x, u) + g(x, u)u (P_k D_x u f(x, u))]ds(u)dx^n. \end{aligned} \tag{1}$$

Since $g(x, u)uD_xP_{k,r}$, $f(x, u)$, $g(x, u)$ and $P_kD_xuf(x, u)$ are monogenic functions in u ,

$$\int_{\mathbb{S}^{n-1}} (g(x, u)uD_xP_{k,r})uf(x, u)ds(u) = 0 = \int_{\mathbb{S}^{n-1}} g(x, u)u(P_kD_xuf(x, u))ds(u).$$

Thus the previous integral (1) equals zero.

By Stokes' Theorem for the Dirac operator, we have

$$\begin{aligned} & \int_{\Omega} [(g(x, u)uD_x, uf(x, u))_u + (g(x, u)u, D_xuf(x, u))_u]dx^n \\ &= \int_{\Omega} \int_{\mathbb{S}^{n-1}} [(g(x, u)uD_x)uf(x, u) + g(x, u)u(D_xuf(x, u))]ds(u)dx^n \\ &= \int_{\partial\Omega} \int_{\mathbb{S}^{n-1}} [(g(x, u)ud\sigma_xuf(x, u))]ds(u) \\ &= \int_{\partial\Omega} (g(x, u)u, d\sigma_xuf(x, u))_u. \end{aligned}$$

But

$$\int_{\partial\Omega} (g(x, u)u, P_kd\sigma_xuf(x, u))_u = \int_{\partial\Omega} \int_{\mathbb{S}^{n-1}} g(x, u)u(P_kd\sigma_xuf(x, u))ds(u) = 0,$$

since $\int_{\mathbb{S}^{n-1}} g(x, u)u(P_kd\sigma_xuf(x, u))ds(u) = 0$.

Therefore we have shown

$$\begin{aligned} & \int_{\Omega} [(g(x, u)uQ_{k,r}, uf(x, u))_u + (g(x, u)u, Q_kuf(x, u))_u]dx^n \\ &= \int_{\partial\Omega} (g(x, u)u, (I - P_k)d\sigma_xuf(x, u))_u. \quad \blacksquare \end{aligned}$$

Remark 1. In the proof of the previous theorem it is proved that

$$\int_{\partial\Omega} (g(x, u)u, (I - P_k)d\sigma_xuf(x, u))_u = \int_{\partial\Omega} (g(x, u)u, d\sigma_xuf(x, u))_u. \quad (2)$$

Theorem 3. (Borel-Pompeiu Theorem) Let Ω' and Ω be as in the previous Theorem. Then for $f \in C^1(\Omega', \mathcal{M}_{k-1})$ and $y \in \Omega$, we obtain

$$\begin{aligned} uf(y, u) &= \int_{\Omega} (H_k(x - y, u, v), Q_kv f(x, v))_v dx^n \\ &\quad - \int_{\partial\Omega} (H_k(x - y, u, v), (I - P_k)d\sigma_v f(x, v))_v. \end{aligned}$$

Here we will use the representation $H_k(x - y, u, v) = \frac{-1}{\omega_n c_k} u Z_{k-1}(u, \frac{(x - y)v(x - y)}{\|x - y\|^2}) \frac{x - y}{\|x - y\|^n} v$.

Proof: Consider a ball $B(y, r)$ centered at y with radius r such that $\overline{B(y, r)} \subset \Omega$. We have

$$\begin{aligned} & \int_{\Omega} (H_k(x - y, u, v), Q_k v f(x, v))_v dx^n \\ &= \int_{\Omega \setminus B(y, r)} (H_k(x - y, u, v), Q_k v f(x, v))_v dx^n \\ &+ \int_{B(y, r)} (H_k(x - y, u, v), Q_k v f(x, v))_v dx^n. \end{aligned}$$

The last integral in the previous equation tends to zero as r tends to zero. This follows from the degree of homogeneity of $x - y$ in $H_k(x - y, u, v)$. Now applying Stokes' Theorem version 2 to the first integral, one gets

$$\begin{aligned} & \int_{\Omega \setminus B(y, r)} (H_k(x - y, u, v), Q_k v f(x, v))_v dx^n \\ &= \int_{\partial \Omega} (H_k(x - y, u, v), (I - P_k) d\sigma_x v f(x, v))_v - \int_{\partial B(y, r)} (H_k(x - y, u, v), (I - P_k) d\sigma_x v f(x, v))_v. \end{aligned}$$

Now let us look at the integral

$$\begin{aligned} & \int_{\partial B(y, r)} (H_k(x - y, u, v), (I - P_k) d\sigma_x v f(x, v))_v dx^n \\ &= \int_{\partial B(y, r)} (H_k(x - y, u, v), (I - P_k) d\sigma_x v f(y, v))_v \\ &+ \int_{\partial B(y, r)} (H_k(x - y, u, v), (I - P_k) d\sigma_x v [f(x, v) - f(y, v)])_v. \end{aligned}$$

Since the second integral on the right hand side tends to zero as r goes to zero because of the continuity of f , we only need to deal with the first integral

$$\begin{aligned} & \int_{\partial B(y, r)} (H_k(x - y, u, v), (I - P_k) d\sigma_x v f(y, v))_v \\ &= \int_{\partial B(y, r)} \int_{\mathbb{S}^{n-1}} H_k(x - y, u, v) (I - P_k) d\sigma_x v f(y, v) ds(v) \\ &= \int_{\partial B(y, r)} \int_{\mathbb{S}^{n-1}} \frac{-1}{\omega_n c_k} u Z_{k-1} \left(u, \frac{(x - y)v(x - y)}{\|x - y\|^2} \right) \frac{x - y}{\|x - y\|^n} v (I - P_k) n(x) v f(y, v) ds(v) d\sigma(x), \end{aligned}$$

where $n(x)$ is the unit outer normal vector and $d\sigma(x)$ is the scalar measure on $\partial B(y, r)$.

Now $n(x)$ here is $\frac{y-x}{\|x-y\|}$. Using equation (2) the previous integral becomes

$$\begin{aligned} & \int_{\partial B(y,r)} \int_{\mathbb{S}^{n-1}} \frac{1}{\omega_n c_k} u Z_{k-1} \left(u, \frac{(x-y)v(x-y)}{\|x-y\|^2} \right) \frac{x-y}{\|x-y\|^n} v \frac{x-y}{\|x-y\|} v f(y, v) ds(v) d\sigma(x) \\ &= \frac{1}{\omega_n c_k} \int_{\partial B(y,r)} \frac{1}{r^{n-1}} \int_{\mathbb{S}^{n-1}} u Z_{k-1} \left(u, \frac{(x-y)v(x-y)}{\|x-y\|^2} \right) \frac{x-y}{\|x-y\|} v \frac{x-y}{\|x-y\|} v f(y, v) ds(v) d\sigma(x) \end{aligned}$$

Since $Z_{k-1} \left(u, \frac{(x-y)v(x-y)}{\|x-y\|^2} \right) \frac{x-y}{\|x-y\|} v \frac{x-y}{\|x-y\|}$ is a harmonic polynomial with degree k in v , we can apply Lemma 5 in [DLRV], then the integral is equal to

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} u Z_{k-1}(u, v) v v f(y, v) ds(v) d\sigma(x) \\ &= -u \int_{\mathbb{S}^{n-1}} Z_{k-1}(u, v) f(y, v) ds(v) = -u f(y, u), \end{aligned}$$

Therefore, when the radius r tends to zero, we obtain the desired result. \blacksquare

Now if the function has compact support in Ω , then by Borel-Pompeiu Theorem we obtain:

Theorem 4. $\iint_{\mathbb{R}^n} (H_k(x-y, u, v), Q_k v \phi(x, v))_v dx^n = u \phi(y, u)$ for each $\phi \in C_0^\infty(\mathbb{R}^n)$.

Now suppose $v f(x, v)$ is a solution to the Q_k operator, then using the Borel-Pompeiu Theorem we have:

Theorem 5. (Cauchy Integral Formula) If $Q_k v f(x, v) = 0$, then for $y \in \Omega$,

$$\begin{aligned} u f(y, u) &= - \int_{\partial \Omega} (H_k(x-y, u, v), (I - P_k) d\sigma_x v f(x, v))_v \\ &= - \int_{\partial \Omega} (H_k(x-y, u, v) d\sigma_x (I - P_{k,r}), v f(x, v))_v. \quad \blacksquare \end{aligned}$$

We also can talk about a Cauchy transform for the Q_k operators:

Definition 2. For a domain $\Omega \subset \mathbb{R}^n$ and a function $f : \Omega \times \mathbb{R}^n \rightarrow Cl_n$, where $f(x, u)$ is monogenic in u with degree $k-1$, the Cauchy (or T_k -transform) of f is formally defined to be

$$(T_k v f)(y, v) = - \iint_{\Omega} (H_k(x-y, u, v), u f(x, u))_u dx^n, \quad y \in \Omega.$$

Theorem 6.

$$Q_k \iint_{\mathbb{R}^n} (H_k(x-y, u, v), v \phi(x, v))_v dx^n = u \phi(y, u), \text{ for } \phi \in C_0^\infty(\mathbb{R}^n).$$

Here we use the representation $H_k(x-y, u, v) = \frac{-1}{\omega_n c_k} u Z_{k-1} \left(u, \frac{(x-y)v(x-y)}{\|x-y\|^2} \right) \frac{x-y}{\|x-y\|^n} v$.

Proof: For each fixed $y \in \mathbb{R}^n$, we can construct a bounded rectangle $R(y)$ centered at y in \mathbb{R}^n .

Then

$$\begin{aligned} & Q_k \iint_{\mathbb{R}^n \setminus R(y)} (H_k(x - y, u, v), v\phi(x, v))_v dx^n \\ &= (I - P_k) D_y \iint_{\mathbb{R}^n \setminus R(y)} (H_k(x - y, u, v), v\phi(x, v))_v dx^n = 0. \end{aligned}$$

Now consider

$$\begin{aligned} & \frac{\partial}{\partial y_i} \iint_{R(y)} (H_k(x - y, u, v), v\phi(x, v))_v dx^n \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[\iint_{R(y)} (H_k(x - y, u, v), v\phi(x, v))_v dx^n \right. \\ & \quad \left. - \iint_{R(y + \varepsilon e_i)} (H_k(x - y - \varepsilon e_i, u, v), v\phi(x, v))_v dx^n \right] \\ &= \iint_{R(y)} (H_k(x - y, u, v), \frac{\partial v\phi(x, v)}{\partial x_i})_v dx^n \\ & \quad + \int_{\partial R_1(y) \cup \partial R_2(y)} (H_k(x - y, u, v), v\phi(x, v))_v d\sigma(x) \end{aligned}$$

where $\partial R_1(y)$ and $\partial R_2(y)$ are the two faces of $R(y)$ with normal vectors $\pm e_i$. So

$$\begin{aligned} & D_y \iint_{R(y)} (H_k(x - y, u, v), v\phi(x, v))_v dx^n \\ &= \iint_{R(y)} \sum_{i=1}^n e_i (H_k(x - y, u, v), \frac{\partial v\phi(x, v)}{\partial x_i})_v dx^n \\ & \quad + \int_{\partial R(y)} n(x) (H_k(x - y, u, v), v\phi(x, v))_v d\sigma(x). \end{aligned}$$

The first integral tends to zero as the volume of $R(y)$ tends to zero. Thus we will pay attention to the integral

$$(I - P_k) \int_{\partial R(y)} n(x) (H_k(x - y, u, v), v\phi(x, v))_v d\sigma(x).$$

This is equal to

$$(I - P_k) \int_{\partial R(y)} \int_{\mathbb{S}^{n-1}} n(x) H_k(x - y, u, v) v\phi(x, v) ds(v) d\sigma(x),$$

which in turn is equal to

$$(I - P_k) \int_{\partial R(y)} \int_{\mathbb{S}^{n-1}} n(x) H_k(x - y, u, v) v \phi(y, v) ds(v) d\sigma(x) \\ + (I - P_k) \int_{\partial R(y)} \int_{\mathbb{S}^{n-1}} n(x) H_k(x - y, u, v) v (\phi(x, v) - \phi(y, v)) ds(v) d\sigma(x).$$

But the last integral on the right side of the above formula tends to zero as the surface area of $\partial R(y)$ tends to zero because of the degree of the homogeneity of $x - y$ in H_k and the continuity of the function ϕ . Hence we are left with

$$(I - P_k) \int_{\partial R(y)} \int_{\mathbb{S}^{n-1}} n(x) H_k(x - y, u, v) v \phi(y, v) ds(v) d\sigma(x).$$

By Stokes' Theorem this is equal to

$$(I - P_k) \int_{\partial B(y, r)} \int_{\mathbb{S}^{n-1}} n(x) H_k(x - y, u, v) v \phi(y, v) ds(v) d\sigma(x).$$

In turn this is equal to

$$(I - P_k) \int_{\partial B(y, r)} \int_{\mathbb{S}^{n-1}} -\frac{x - y}{\|x - y\|} \frac{-1}{\omega_n c_k} u Z_{k-1}(u, \frac{(x - y)v(x - y)}{\|x - y\|^2}) \frac{x - y}{\|x - y\|^n} v v \phi(y, v) ds(v) d\sigma(x) \\ = (I - P_k) \int_{\partial B(y, r)} \frac{-1}{\omega_n c_k} \int_{\mathbb{S}^{n-1}} \frac{x - y}{\|x - y\|} u Z_{k-1}(u, \frac{(x - y)v(x - y)}{\|x - y\|^2}, v) \frac{x - y}{\|x - y\|^n} \phi(y, v) ds(v) d\sigma(x).$$

Since $Z_{k-1}(u, v)$ is the reproducing kernel of \mathcal{M}_{k-1} , $\pm \tilde{a} Z_{k-1}(au\tilde{a}, av\tilde{a})a$ is also the reproducing kernel of \mathcal{M}_{k-1} for each $a \in Pin(n)$. See [DLRV]. Now let $a = \frac{x - y}{\|x - y\|}$, the previous integral equals

$$(I - P_k) \int_{\partial B(y, r)} \frac{1}{\omega_n c_k} \int_{\mathbb{S}^{n-1}} \frac{x - y}{\|x - y\|} u \frac{x - y}{\|x - y\|} Z_{k-1}(\frac{(x - y)u(x - y)}{\|x - y\|^2}, v) \frac{x - y}{\|x - y\|} \frac{x - y}{\|x - y\|^n} \\ \phi(y, v) ds(v) d\sigma(x) \\ = (I - P_k) \int_{\partial B(y, r)} \int_{\mathbb{S}^{n-1}} \frac{1}{\omega_n c_k} \frac{1}{r^{n-1}} \frac{x - y}{\|x - y\|} u \frac{x - y}{\|x - y\|} Z_{k-1}(\frac{(x - y)u(x - y)}{\|x - y\|^2}, v) \phi(y, v) ds(v) d\sigma(x).$$

Applying Lemma 5 in [DLRV], the integral becomes

$$(I - P_k) \int_{\mathbb{S}^{n-1}} u Z_{k-1}(u, v) \phi(y, v) ds(v) = (I - P_k) u \phi(y, u) = u \phi(y, u). \quad \blacksquare$$

5 The Q_k operators on the sphere

In this section, we will extend the results for the Q_k operators in \mathbb{R}^n from the previous sections to the sphere.

Consider the Cayley transformation $C : \mathbb{R}^n \rightarrow \mathbb{S}^n$, where \mathbb{S}^n is the unit sphere in \mathbb{R}^{n+1} , defined by $C(x) = (e_{n+1}x + 1)(x + e_{n+1})^{-1}$, where $x = x_1e_1 + \dots + x_ne_n \in \mathbb{R}^n$, and e_{n+1} is a unit vector in \mathbb{R}^{n+1} which is orthogonal to \mathbb{R}^n . Now $C(\mathbb{R}^n) = \mathbb{S}^n \setminus \{e_{n+1}\}$. Suppose $x_s \in \mathbb{S}^n$ and $x_s = x_{s_1}e_1 + \dots + x_{s_n}e_n + x_{s_{n+1}}e_{n+1}$, then we have $C^{-1}(x_s) = (-e_{n+1}x_s + 1)(x_s - e_{n+1})^{-1}$.

The Dirac operator over the n -sphere \mathbb{S}^n has the form $D_s = w(\Gamma + \frac{n}{2})$, where $w \in \mathbb{S}^n$ and $\Gamma = \sum_{i < j, i=1}^n e_ie_j(w_i \frac{\partial}{\partial w_j} - w_j \frac{\partial}{\partial w_i})$. See [CM, LR, R1, R2, Va3].

Let U be a domain in \mathbb{R}^n . Consider a function $f_* : U \times \mathbb{R}^n \rightarrow Cl_{n+1}$ such that for each $x \in U$, $f_*(x, u)$ is a left monogenic polynomial homogeneous of degree $k - 1$ in u . This function reduces to $f(x_s, u)$ on $C(U) \times \mathbb{R}^n$ and $f(x_s, u)$ takes its values in Cl_{n+1} , where $C(U) \subset \mathbb{S}^n$ and $f(x_s, u)$ is a left monogenic polynomial homogeneous of degree $k - 1$ in u .

The left n -spherical remaining operator on the sphere is defined to be

$$Q_k^S =: (I - P_k)D_{s, x_s},$$

where D_{s, x_s} is the Dirac operator on the sphere with respect to x_s .

Hence the left n -spherical remaining equation is defined to be

$$Q_k^S u f(x_s, u) = 0.$$

On the other hand, the right n -spherical remaining operator is defined to be

$$Q_{k,r}^S := D_{s, x_s}(I - P_{k,r}).$$

The right n -spherical remaining equation is defined to be $g(x_s, v)vQ_{k,r}^S = 0$, where $g(x_s, v) \in \overline{\mathcal{M}}_{k-1}$.

5.1 The intertwining operators for Q_k^S and Q_k operators and the conformal invariance of $Q_k^S u f(x_s, u) = 0$

First let us recall that if $f(u) \in \mathcal{M}_{k-1}$ then it trivially extends to $F(v) = f(u + u_{n+1}e_{n+1})$ with $u_{n+1} \in \mathbb{R}$ and $F(v) = f(u)$ for all $u_{n+1} \in \mathbb{R}$. Consequently $D_{n+1}F(v) = 0$ where

$$D_{n+1} = \sum_{j=1}^{n+1} e_j \frac{\partial}{\partial u_j}.$$

If $f(u) \in \mathcal{M}_{k-1}$ then for any boundary of a piecewise smooth bounded domain $U \subseteq \mathbb{R}^n$ by Cauchy's Theorem

$$\int_{\partial U} n(u) f(u) d\sigma(u) = 0. \quad (3)$$

Suppose now $a \in \text{Pin}(n+1)$ and $u = aw\tilde{a}$ then although $u \in \mathbb{R}^n$ in general w belongs to the hyperplane $a^{-1}\mathbb{R}^n\tilde{a}^{-1}$ in \mathbb{R}^{n+1} .

By applying a change of variable, up to a sign the integral (3) becomes

$$\int_{a^{-1}\partial U\tilde{a}^{-1}} an(w)\tilde{a}F(aw\tilde{a})d\sigma(w) = 0. \quad (4)$$

As ∂U is arbitrary then on applying Stokes' Theorem to (4) we see that

$$D_a \tilde{a} F(aw\tilde{a}) = 0, \quad \text{where } D_a := D_{n+1} \big|_{a^{-1}\mathbb{R}^n \tilde{a}^{-1}}.$$

See [LRV].

From now on all functions on spheres take their values in Cl_{n+1} .

Now let $f(x_s, u) : U_s \times \mathbb{R}^n \rightarrow Cl_{n+1}$ be a monogenic polynomial homogeneous of degree k in u for each $x_s \in U_s$, where U_s is a domain in \mathbb{S}^n .

It is known from section 3 that $I - P_k$ is conformally invariant under a general Möbius transformation over \mathbb{R}^n . This trivially extends to Möbius transformations on \mathbb{R}^{n+1} . It follows that if we restrict x_s to \mathbb{S}^n , then $I - P_k$ is also conformally invariant under the Cayley transformation C and its inverse C^{-1} , with $x \in \mathbb{R}^n$.

We can use the intertwining formulas for D_x and D_{s,x_s} given in [LR] to establish the intertwining formulas for Q_k and Q_k^S .

Theorem 7.

$$-J_{-1}(C^{-1}, x_s) Q_{k,u} u f(x, u) = Q_{k,w}^S w J(C^{-1}, x_s) f(C^{-1}(x_s), \frac{(x_s - e_{n+1})w(x_s - e_{n+1})}{\|x_s - e_{n+1}\|^2}),$$

where $Q_{k,u}$ are the remaining operators with respect to $u \in \mathbb{R}^n$, $Q_{k,w}^S$ are the remaining operators on the sphere with respect to $w \in \mathbb{S}^n$, $u = \frac{(x_s - e_{n+1})w(x_s - e_{n+1})}{\|x_s - e_{n+1}\|^2}$, $J(C^{-1}, x_s) = \frac{x_s - e_{n+1}}{\|x_s - e_{n+1}\|^n}$ is the conformal weight for the inverse of the Cayley transformation and $J_{-1}(C^{-1}, x_s) = \frac{x_s - e_{n+1}}{\|x_s - e_{n+1}\|^{n+2}}$.

Proof: In [LR] it is shown that $D_x = J_{-1}(C^{-1}, x_s)^{-1} D_{s,x_s} J(C^{-1}, x_s)$.

Set $u = \frac{J(C^{-1}, x_s)wJ(C^{-1}, x_s)}{\|J(C^{-1}, x_s)\|^2}$ for some $w \in \mathbb{R}^{n+1}$. Consequently,

$$\begin{aligned} Q_{k,u} u f(x, u) &= (I - P_{k,u}) D_x u f(x, u) \\ &= (I - P_{k,u}) J_{-1}(C^{-1}, x_s)^{-1} D_{s,x_s} J(C^{-1}, x_s) u f(C^{-1}(x_s), u) \\ &= J_{-1}(C^{-1}, x_s)^{-1} (I - P_{k,w}) D_{s,x_s} J(C^{-1}, x_s) \frac{J(C^{-1}, x_s)wJ(C^{-1}, x_s)}{\|J(C^{-1}, x_s)\|^2} \\ &\quad f(C^{-1}(x_s), \frac{J(C^{-1}, x_s)wJ(C^{-1}, x_s)}{\|J(C^{-1}, x_s)\|^2}) \end{aligned}$$

Since $\frac{J(C^{-1}, x_s)wJ(C^{-1}, x_s)}{\|J(C^{-1}, x_s)\|^2} = \frac{(x_s - e_{n+1})w(x_s - e_{n+1})}{\|x_s - e_{n+1}\|^2}$, the previous equation becomes

$$Q_{k,u} u f(x, u) = -J_{-1}(C^{-1}, x_s)^{-1} Q_{k,w}^S w J(C^{-1}, x_s) f(C^{-1}(x_s), \frac{(x_s - e_{n+1})w(x_s - e_{n+1})}{\|(x_s - e_{n+1})\|^2}). \quad \blacksquare$$

Similarly, we have the following result for the remaining operators under the Cayley transformation.

Theorem 8.

$$-J_{-1}(C, x)Q_{k,u}^S u g(x_s, u) = Q_{k,w} w J(C, x) g(C(x), \frac{(x + e_{n+1})w(x + e_{n+1})}{\|x + e_{n+1}\|^2}),$$

where $u = \frac{(x + e_{n+1})w(x + e_{n+1})}{\|x + e_{n+1}\|^2}$, $J(C, x) = \frac{x + e_{n+1}}{\|x + e_{n+1}\|^n}$ and $J_{-1}(C, x) = \frac{x + e_{n+1}}{\|x + e_{n+1}\|^{n+2}}$ is the conformal weight for the Cayley transformation.

As a consequence of two previous theorems we have the conformal invariance of equation $Q_k^S u f(x_s, u) = 0$:

Theorem 9. $Q_{k,u}^S u f(x, u) = 0$ if and only if

$$Q_{k,w}^S w J(C^{-1}, x_s) f(C^{-1}(x_s), \frac{(x_s - e_{n+1})w(x_s - e_{n+1})}{\|x_s - e_{n+1}\|^2}) = 0$$

and $Q_{k,u}^S u g(x_s, u) = 0$ if and only if

$$Q_{k,w} w J(C, x) g(C(x), \frac{(x + e_{n+1})w(x + e_{n+1})}{\|x + e_{n+1}\|^2}) = 0.$$

5.2 A kernel for the Q_k^S operator

Now consider the kernel in \mathbb{R}^n

$$\begin{aligned} & \frac{-1}{\omega_n c_k} w \frac{x - y}{\|x - y\|^n} Z_{k-1}(\frac{(x - y)w(x - y)}{\|x - y\|^2}, v) v \\ &= \frac{-1}{\omega_n c_k} \frac{J(C^{-1}, x_s)^{-1} u J(C^{-1}, x_s)^{-1}}{\|J(C^{-1}, x_s)^{-1}\|^2} \\ & J(C^{-1}, x_s)^{-1} \frac{x_s - y_s}{\|x_s - y_s\|^n} J(C^{-1}, y_s)^{-1} Z_{k-1}(\frac{(x - y)w(x - y)}{\|x - y\|^2}, v) v, \end{aligned}$$

where $w = \frac{J(C^{-1}, x_s)^{-1} u J(C^{-1}, x_s)^{-1}}{\|J(C^{-1}, x_s)^{-1}\|^2}$.

Multiplying by $J(C^{-1}, x_s)$ and applying the Cayley transformation to the above kernel, we obtain the kernel

$$H_k^S(x - y, u, v) := \frac{-1}{\omega_n c_k} u \frac{x_s - y_s}{\|x_s - y_s\|^n} J(C^{-1}, y_s)^{-1} Z_{k-1}(au\tilde{a}, v) v, \quad (5)$$

where $a = a(x_s, y_s) = \frac{J(C^{-1}, x_s)^{-1}(x_s - y_s)J(C^{-1}, y_s)^{-1}}{\|J(C^{-1}, x_s)^{-1}\|\|x_s - y_s\|\|J(C^{-1}, y_s)^{-1}\|}$.

This is a fundamental solution to $Q_k^S u f(x_s, u) = 0$ on \mathbb{S}^n , for $x_s, y_s \in \mathbb{S}^n$.

Similarly, we obtain that

$$\frac{-1}{\omega_n c_k} u Z_{k-1}(u, \tilde{a} v a) J(C^{-1}, y_s)^{-1} \frac{x_s - y_s}{\|x_s - y_s\|^n} v \quad (6)$$

is a non trivial solution to $g(x_s, v)vQ_{k,r}^S = 0$.

We can see that the representations (5) and (6) are the same up to a reflection by

$$\begin{aligned}
& \frac{-1}{\omega_n c_k} u Z_{k-1}(u, \tilde{a} v a) J(C^{-1}, y_s)^{-1} \frac{x_s - y_s}{\|x_s - y_s\|^n} v \\
&= \frac{1}{\omega_n c_k} u \tilde{a} Z_k(a u \tilde{a}, v) a J(C^{-1}, y_s)^{-1} \frac{x_s - y_s}{\|x_s - y_s\|^n} v \\
&= \frac{-1}{\omega_n c_k} u J(C^{-1}, y_s)^{-1} \frac{x_s - y_s}{\|x_s - y_s\|^n} \frac{J(C^{-1}, x_s)^{-1}}{\|J(C^{-1}, x_s)^{-1}\|} Z_k(a u \tilde{a}, v) \frac{J(C^{-1}, x_s)^{-1}}{\|J(C^{-1}, x_s)^{-1}\|} v \\
&= u \frac{J(C^{-1}, x_s)^{-1}}{\|J(C^{-1}, x_s)^{-1}\|} \frac{-1}{\omega_n c_k} \frac{x_s - y_s}{\|x_s - y_s\|^n} J(C^{-1}, y_s)^{-1} Z_k(a u \tilde{a}, v) \frac{J(C^{-1}, x_s)^{-1}}{\|J(C^{-1}, x_s)^{-1}\|} v.
\end{aligned}$$

5.3 Some basic integral formulas for the remaining operators on spheres

In this section we will study some basic integral formulas related to the remaining operators on the sphere.

Theorem 10. (*Stokes' Theorem for the n -spherical Dirac operator D_s*) [LR]

Suppose U_s is a domain on \mathbb{S}^n and $f, g : U_s \times \mathbb{R}^n \rightarrow Cl_{n+1}$ are C^1 , then for a subdomain V_s of U_s , we have

$$\begin{aligned}
& \int_{\partial V_s} g(x_s, u) n(x_s) f(x_s, u) d\Sigma(x_s) \\
&= \int_{V_s} (g(x_s, u) D_{s, x_s} f(x_s, u) + g(x_s, u) (D_{s, x_s} f(x_s, u)) dS(x_s),
\end{aligned}$$

where $dS(x_s)$ is the n -dimensional area measure on V_s , $d\Sigma(x_s)$ is the $n - 1$ -dimensional scalar Lebesgue measure on ∂V_s and $n(x_s)$ is the unit outward normal vector to ∂V_s at x_s .

Applying the similar arguments to prove the Stokes' Theorem for Q_k operators in section 4, we can obtain

Theorem 11. (*Stokes' Theorem for the Q_k^S operator*)

Let $U_s, V_s, \partial V_s$ be as in the previous Theorem. Then for $f, g \in C^1(U_s \times \mathbb{R}^n, \mathcal{M}_k)$, we have version 1

$$\begin{aligned}
& \int_{V_s} [(g(x_s, u) Q_{k,r}^S f(x_s, u))_u + (g(x_s, u), Q_k^S f(x_s, u))_u] dS(x_s) \\
&= \int_{\partial V_s} (g(x_s, u), (I - P_k) n(x_s) f(x_s, u))_u d\Sigma(x) \\
&= \int_{\partial V_s} (g(x_s, u) n(x_s) (I - P_{k,r}), f(x_s, u))_u d\Sigma(x).
\end{aligned}$$

Then for $f, g \in C^1(U_s \times \mathbb{R}^n, \mathcal{M}_{k-1})$, we have version 2

$$\begin{aligned}
& \int_{V_s} [(g(x_s, u)u Q_{k,r}^S, uf(x_s, u))_u + (g(x_s, u)u, Q_k^S uf(x_s, u))_u] dS(x_s) \\
&= \int_{\partial V_s} (g(x_s, u)u, (I - P_k)n(x_s)uf(x_s, u))_u d\Sigma(x) \\
&= \int_{\partial V_s} (g(x_s, u)un(x_s)(I - P_{k,r}), uf(x_s, u))_u d\Sigma(x).
\end{aligned}$$

Remark 2. Using the similar arguments to show the conformal invariance of Stokes' Theorem for the Rarita-Schwinger operators in [LRV], we obtain that Stokes' Theorem for the Q_k operators is conformally invariant under the Cayley transformation and the inverse of the Cayley transformation.

Remark 3. We also have the following fact

$$\int_{\partial V_s} (g(x_s, u)u, (I - P_k)n(x_s)uf(x_s, u))_u d\Sigma(x) = \int_{\partial V_s} (g(x_s, u)u, n(x_s)uf(x_s, u))_u d\Sigma(x) \quad (7)$$

Theorem 12. (Borel-Pompeiu Theorem) Suppose U_s , V_s and ∂V_s are stated as in Theorem 10 and $y_s \in V_s$. Then for $f \in C^1(U_s \times \mathbb{R}^n, \mathcal{M}_{k-1})$ we have

$$\begin{aligned}
u'f(y_s, u') &= J(C^{-1}, y_s) \int_{\partial V_s} (H_k^S(x_s - y_s, u, v), (I - P_k)n(x_s)vf(x_s, v))_v d\Sigma(x_s) \\
&\quad - J(C^{-1}, y_s) \int_{V_s} (H_k^S(x_s - y_s, u, v), Q_k^S vf(x_s, v))_v dS(x_s)
\end{aligned}$$

where $u' = \frac{(y_s - e_{n+1})u(y_s - e_{n+1})}{\|y_s - e_{n+1}\|^2}$, $dS(x_s)$ is the n -dimensional area measure on $V_s \subset \mathbb{S}^n$, $n(x_s)$ and $d\Sigma(x_s)$ as in Theorem 10.

Proof: In the proof we use the representation

$$H_k^S(x - y, u, v) = \frac{-1}{\omega_n c_k} u Z_{k-1}(u, \tilde{a}va) J(C^{-1}, y_s)^{-1} \frac{x_s - y_s}{\|x_s - y_s\|^n} v.$$

Let $B_s(y_s, \epsilon)$ be the ball centered at $y_s \in \mathbb{S}^n$ with radius ϵ . We denote $C^{-1}(B_s(y_s, \epsilon))$ by $B(y, r)$, and $C^{-1}(\partial B_s(y_s, \epsilon))$ by $\partial B(y, r)$, where $y = C^{-1}(y_s) \in \mathbb{R}^n$ and r is the radius of $B(y, r)$ in \mathbb{R}^n . Using the similar arguments in the proof of Theorem 2, we only deal with

$$\begin{aligned}
& \int_{\partial B_s(y_s, \epsilon)} (H_k^S(x_s - y_s, u, v), (I - P_k)n(x_s)vf(y_s, v))_v d\Sigma(x_s) \\
&= \int_{\partial B_s(y_s, \epsilon)} \int_{\mathbb{S}^{n-1}} H_k^S(x_s - y_s, u, v) (I - P_k)n(x_s)vf(y_s, v) ds(v) d\Sigma(x_s).
\end{aligned}$$

Now applying (7), the integral is equal to

$$\begin{aligned}
& \int_{\partial B_s(y_s, \epsilon)} \int_{\mathbb{S}^{n-1}} H_k^S(x_s - y_s, u, v) n(x_s) v f(y_s, v) ds(v) d\Sigma(x_s) \\
&= \int_{\partial B_s(y_s, \epsilon)} \int_{\mathbb{S}^{n-1}} \frac{-1}{\omega_n c_k} u Z_{k-1}(u, \tilde{a} v a) J(C^{-1}, y_s)^{-1} \frac{x_s - y_s}{\|x_s - y_s\|^n} v n(x_s) v f(y_s, v) ds(v) d\Sigma(x_s) \\
&\quad \text{Applying the inverse of the Cayley transformation to the previous integral, we have} \\
&= \int_{\partial B(y, r)} \int_{\mathbb{S}^{n-1}} \frac{-1}{\omega_n c_k} u Z_{k-1}(u, \frac{(x-y)w(x-y)}{\|x-y\|^2}) J(C^{-1}, y_s)^{-1} J(C, y)^{-1} \frac{x-y}{\|x-y\|^n} J(C, x)^{-1} \\
&\quad v J(C, x) n(x) J(C, x) v f(C(y), \frac{J(C, y)wJ(C, y)}{\|J(C, y)\|^2}) ds(v) d\sigma(x),
\end{aligned}$$

where $v = \frac{J(C, y)wJ(C, y)}{\|J(C, y)\|^2}$.

Now if we replace v with $\frac{J(C, y)wJ(C, y)}{\|J(C, y)\|^2}$ in the previous integral and we also set $J(C, x) = (J(C, x) - J(C, y)) + J(C, y)$, but $J(C, x) - J(C, y)$ tends to zero as x approaches y . Thus the previous integral can be replaced by

$$\begin{aligned}
&= \int_{\partial B(y, r)} \int_{\mathbb{S}^{n-1}} \frac{-1}{\omega_n c_k} u Z_{k-1}(u, \frac{(x-y)w(x-y)}{\|x-y\|^2}) \frac{x-y}{\|x-y\|^n} J(C, y)^{-1} \\
&\quad \frac{J(C, y)wJ(C, y)}{\|J(C, y)\|^2} J(C, y) n(x) J(C, y) \frac{J(C, y)wJ(C, y)}{\|J(C, y)\|^2} f(C(y), \frac{J(C, y)wJ(C, y)}{\|J(C, y)\|^2}) ds(w) d\sigma(x) \\
&= \int_{\partial B(y, r)} \int_{\mathbb{S}^{n-1}} \frac{-1}{\omega_n c_k} u Z_{k-1}(u, \frac{(x-y)w(x-y)}{\|x-y\|^2}) \frac{x-y}{\|x-y\|^n} w n(x) \\
&\quad w J(C, y) f(C(y), \frac{J(C, y)wJ(C, y)}{\|J(C, y)\|^2}) ds(w) d\sigma(x) \\
&= \int_{\partial B(y, r)} \int_{\mathbb{S}^{n-1}} \frac{1}{\omega_n c_k} \frac{1}{r^{n-1}} u Z_{k-1}(u, \frac{(x-y)w(x-y)}{\|x-y\|^2}) \frac{x-y}{\|x-y\|} w \frac{x-y}{\|x-y\|} \\
&\quad w J(C, y) f(C(y), \frac{J(C, y)wJ(C, y)}{\|J(C, y)\|^2}) ds(w) d\sigma(x).
\end{aligned}$$

Using Lemma 5 in [DLRV], the previous integral becomes

$$\begin{aligned}
& \int_{\mathbb{S}^{n-1}} u Z_{k-1}(u, w) w w J(C, y) f(C(y), \frac{J(C, y)wJ(C, y)}{\|J(C, y)\|^2}) ds(w) \\
&= - \int_{\mathbb{S}^{n-1}} u Z_{k-1}(u, w) J(C, y) f(C(y), \frac{J(C, y)wJ(C, y)}{\|J(C, y)\|^2}) ds(w) \\
&= -u J(C, y) f(C(y), \frac{J(C, y)uJ(C, y)}{\|J(C, y)\|^2}). \tag{8}
\end{aligned}$$

If we set $u' = \frac{J(C, y)uJ(C, y)}{\|J(C, y)\|^2} = \frac{J(C^{-1}, y_s)^{-1}uJ(C^{-1}, y_s)^{-1}}{\|J(C^{-1}, y_s)^{-1}\|^2} = \frac{(y_s - e_{n+1})u(y_s - e_{n+1})}{\|y_s - e_{n+1}\|^2}$,

then

$uJ(C, y) = uJ(C^{-1}, y_s)^{-1} = J(C^{-1}, y_s)\|J(C^{-1}, y_s)^{-1}\|^2 u'$. Now if we multiply the both sides of equation (8) by $\frac{J(C^{-1}, y_s)^{-1}}{\|J(C^{-1}, y_s)^{-1}\|^2} = -J(C^{-1}, y_s)$, then we obtain

$$\begin{aligned} & J(C^{-1}, y_s) \int_{\mathbb{S}^{n-1}} u Z_{k-1}(u, w) J(C, y) f(C(y), \frac{J(C, y)wJ(C, y)}{\|J(C, y)\|^2}) ds(w) \\ &= u' f(C(y), u') = u' f(y_s, u'). \quad \blacksquare \end{aligned}$$

Corollary 1. *Let ψ be a function in $C^\infty(V_s, \mathcal{M}_{k-1})$ and $\text{supp } f \subset V_s$. Then*

$$u' \psi(y_s, u') = -J(C^{-1}, y_s) \int_{V_s} (H_k^S(x_s - y_s, u, v), Q_k^S v \psi(x_s, v))_v dS(x_s),$$

where $u' = \frac{(y_s - e_{n+1})u(y_s - e_{n+1})}{\|y_s - e_{n+1}\|^2}$.

Corollary 2. *(Cauchy Integral Formula for Q_k^S operators)*

If $Q_k^S v f(x_s, v) = 0$, then for $y_s \in V_s$ we have

$$\begin{aligned} u' f(y_s, u') &= J(C^{-1}, y_s) \int_{\partial V_s} (H_k^S(x_s - y_s, u, v), (I - P_k)n(x_s)vf(x_s, v))_v d\Sigma(x_s) \\ &= J(C^{-1}, y_s) \int_{\partial V_s} (H_k^S(x_s - y_s, u, v)n(x_s)(I - P_{k,r}), vf(x_s, v))_v d\Sigma(x_s), \end{aligned}$$

where $u' = \frac{(y_s - e_{n+1})u(y_s - e_{n+1})}{\|y_s - e_{n+1}\|^2}$.

Remark 4. *By factoring out \mathbb{S}^n by the group $\mathbb{Z}_2 = \{\pm 1\}$ we obtain real projective space, \mathbb{RP}^n . Using the similar arguments to obtain the results for Rarita-Schwinger operators on real projective space in [LRV], we can easily extend the similar results for Q_k operators to real projective space.*

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